Mathematical Modeling and Analysis

Los Alamos
NATIONAL LABORATORY

Nonlinear, Dispersive Partial Differential Equations

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Introduction

The motion of waves in water and other fluids has given rise to many nonlinear, partial differential equations. Perhaps most famous is the Korteweg-de Vries equation

$$u_t + u_x - \delta u u_x - \beta u_{xxx} = 0 \qquad (1)$$

which describes the propagation of waves in shallow water.

The effect of the dispersive term βu_{xxx} is to broaden the wave, while the nonlinear term $\alpha u u_x$ causes the wave to steepen. Competition between these two effects allows the KdV equation to support solitary wave solutions, or robust nonlinear waves that can retain their form for long time scales, despite interference.

The Benjamin Equation

My research at Los Alamos this summer focused on the dynamics of solutions to another nonlinear, dispersive equation. The Benjamin equation

$$u_t + u_x + \delta u u_x - \alpha L u_x - \beta u_{xxx} = 0 \qquad (2)$$

(here $L = H \partial_x$, and H is the Hilbert transform) describes the vertical displacement of the interface between a thin layer of fluid atop a much thicker layer of higher density fluid, and bounded above and below by rigid horizontal planes.

As with the KdV equation, the combination of nonlinear and dispersive terms (in this case, both the α and β terms are dispersive) admits the possibility of solitary wave solutions. For the Benjamin equation, such solutions require

the right balance between the competing dispersive terms. During the summer, I used a variable order, variable time-step Adams-Bashforth-Moulton method to integrate the equation in time. I studied the behavior of perturbed steady state solutions, as well as compact cos^2 pulses when evolved in time.

In general, it is not possible to write down an analytic solution to the complete Benjamin Equation. However, for $\alpha = 0$ the equation becomes the KdV equation, while setting $\beta = 0$ gives us the Benjamin-Ono equation. Both these equations have known analytic solutions, which we used to validate our method and integrator.
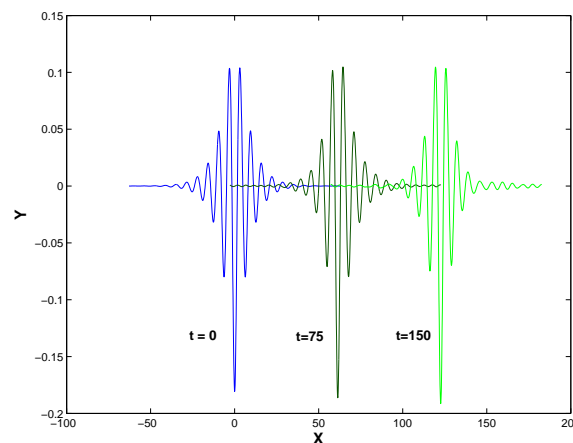
A steady state solution to the Benjamin equation for $\delta = 2$, $\alpha = 1.98$ and $\beta = 1$ is evolved in time using perturbed dispersion coefficient $\alpha = 1.96$. The solution sheds radiation and appears to remain stable.

0.1 Steady State Solutions

A numerical method for generating steady-state solutions to the Benjamin equation is described in [1] and yields solitary wave solutions with oscillatory tails. The solutions we used were generated for fixed $\delta = 2$ and $\beta = 1$, while α varied between 0 and 1.98. In general, the larger the value of α, the more oscillatory the steady-state solution, and the smaller the maximum peak amplitude. Before testing the stability of perturbed solutions, we verified that the unperturbed steady

state solutions in fact showed no change when we used our program to evolve them in time.

To test the stability of these solutions, we perturbed the amplitude, width or α values of the steady state solutions. The effect of perturbing the value of α is to change the balance between the two dispersive terms in the equation.

The perturbed solutions responded in one of three ways when integrated in time. As predicted in [1], we observed that the steady state solutions remained stable for tiny perturbations (smaller than 1%). For larger perturbations, the solutions appeared to transition to another state, and then stop changing. We are currently studying whether these are new steady states. Finally, sufficiently large perturbations caused the solutions to break up entirely.
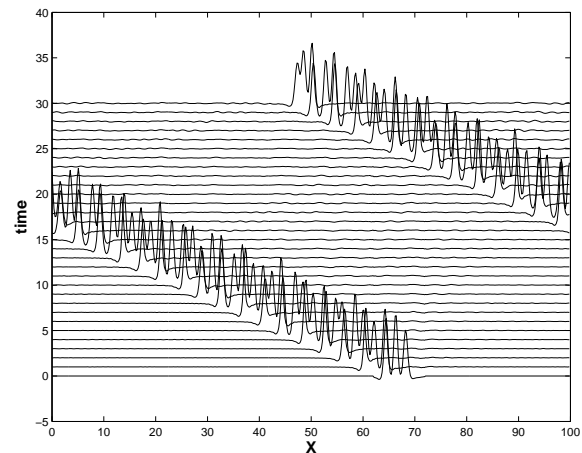
0.2 Multiplet Solutions

Under certain conditions, we observed that compact $\cos^2$ pulses evolved into multi-humped solitary waves with pulsating peaks. Such solutions are called *multiplets*. At present, we have observed these solutions only for values of δ, α and β near 10, 1.3, and 0.5 respectively. This reflects a larger influence from the nonlinearity than present for the steady state solutions, where $\delta = 2$.

Multiplets have been observed as solutions of other nonlinear, dispersive equations. The behavior of multiplet solutions of two such equations is described in [2], where it is observed that multiplets collide nearly elastically and are difficult to fuse together. In future research we will check whether such behavior occurs also for the Benjamin equation.

Conclusions

Steady state solutions to the Benjamin equation appear to remain stable to a small perturbation of amplitude, width, or dispersive coefficients. I am currently using numerical tools I created while at Los Alamos to study the relative error of time integrated perturbed steady solutions as a function of perturbation size. One of our goals is to characterized this relationship and to approximate with high accuracy the perturbation size for which the solutions become unstable.

The space-time evolution of a triplet solution of the Benjamin equation for $\delta = 10$, $\alpha = 1.39$, $\beta = 0.49$. The three peaks of this solitary wave pulsate, yet remain bound to one another.

Multiplets are a little understood type of solution to certain nonlinear, dispersive PDE's, including the Benjamin equation. Further research will be to experiment with the conditions under which such solutions arise, as well as the behavior of interacting multiplets.

Acknowledgements

References

[2] Hyman, J. M. and Rosenau, P., Pulsating multiplet solutions of quintic wave equations, Physica D, 123(1998) 502-512.

[1] Albert, J.P., Bona, J.B.,and Restepo, J.M, Solitary-Wave Solutions of the Benjamin Equation, SIAM J. Appl. Math, vol. 59, no. 6 (1999) 2139-2161.